**Special Relativity Dynamics**

Now we’ll look at a relativistically correct formulation of dynamics.

**Newton’s 2nd law modified to accommodate special relativity**

First we’ll consider Newton’s second law. As originall stated, we have:



This seems to indicate that constant application of a force can accelerate an object up to any velocity. But we know now that c is ultimate speed limit. Therefore these equations cannot be correct, or rather it cannot be the case that p = mv. We *can* keep this form actually, we just have to use the correct ‘time’. We should use the proper time of the object, τ, rather than the time of the rest frame. So then we’d have:



Changing the definition of momentum then, we can rewrite this as the new relativistically correct version of N2L.



We’ll note that as v → 0, γ → 1 and so this definition reduces to the classical one at relativistically small speeds. Let’s play around with this formula a bit,



which gives us a relatively nice form, properly interpreted.



and so we can explicitly see that as the speed of the object decreases to 0, and consequently γ → 1, this equation reduces to the usual version of N2L.

**Special case of circular motion**

Observe that the correction to the expected result is of smaller order than the first term. But in general, the force will be directed along **a** and **v**. Two special cases. If **v** is perpendicular to **a**, like in circular motion, then we get,



**Special case of linear motion**

If **v** is parallel to **a**, like with straight line motion, then we get,



**Example: magnetic field**

We can apply the second expression to the case of charges rotating in a magnetic field,



and so,



as before, but with p redefined.

**Example: Synchotron**

A synchotron produces radiation by confining beams of charged particles with a magnetic field. Suppose the radius of their orbit is fixed to r, for practical purposes, and the maximum confining magnetic field strength is B. If a synchotron can circulate protons with a maximum total energy 3 GeV, what is the maximum kinetic energy would 6+ 14N ions have? Note a proton’s rest mass is m = 938MeV/c2.

We start by finding the momentum of the protons. We’ll put energies in GeV.



Now the proton and nitrogen ion beams have the same radius and field strength, so:



The kinetic energy of our nitrogen ion would then be:



**Example: constant force**

Consider the motion of a particle under a constant force, in 1D, starting from rest. Then we have,



These equations can be integrated.



Solving for v,



which can also be integrated to get position. We get,



**Thomas Precession Modification to Torque Equation**

So the equation:



appears to hold, now, in frames where the particle is not moving, but not in frames where it is. And this is a consequence of Thomas Precesion. When the particle is moving in a frame, this introduces an extra rotation due to the precession and our equation is modified to:



This is extra term is kind of ad hoc justified by the equation below:



And identifying d**Q**/dt)P with the torque , even though it’d seem to me rather that d**Q**/dt should be identified with , since **Q** is in the (perhaps) inertial frame. Whatever.

**WE equation modified to accommodate special relativity**

We also need to modify the work-energy equation. According to Physics 1,



But given a large enough amount of work, this would again imply that we can move the particle up to a velocity greater than c. So what is the correct expression for the energy of a moving particle? This can be deduced from the definition of **p**. Consider,



and so we can say that, separating the work done into work done by conservative and non-conservative forces:



This is our new relativistically correct work-energy equation. It is convenient to note that we can also write the energy as:



which you may verify. Okay well let’s do it,



Let’s examine what E is when v = 0. In that case we have,



This is called the rest energy is the energy. On the other hand, if a particle has no mass, but momentum (how something could have momentum but no mass might be a mystery, but photons have this property), then we’d have,



Incidentally, we can define the KE as the difference between E and E0 – the energy purely due to motion. And so we’d have,



**Example**

Suppose we accelerate a proton through a potential difference of 10MV. What will be its velocity?



and so we get,



Note that for an electron,



and proton,



So this would translate to:



**Relativistically invariant formulation of N2L + WE**

Now we’d like to write these equations out in a relativistically invariant way, meaning we want to write these equations in terms of space-time vectors. So consider:



We can write these two equations as one space-time vector equation as follows:



So then defining the space-time force as:



we can combine the work-energy equation and N2L into a single relativistically invariant equation:



where τ is the time measured in the particles rest frame as always. Be careful to note that this expression is equivalent to the former two. There is nothing wrong with those two and we can use them or this one at our leisure. This is just another way of writing it. Should be noted though, that this implies that the power/force P, **F** are not invariant in reference frames per se´. So if the power/force on a particle is measured to be P/**F** in one frame, that won’t mean that it will be measured to be P/**F** in another frame. Rather, the force transformation law would look like this:



of course. For example, let’s say that we have a force **F** acting on a particle in its instantaneous rest frame. What would be the force acting on it in a frame moving with velocity -**v** (so that the particle is moving with velocity **v** now)? Well, the four-force will transform as (β = v/c):



Can write this as:



And since the four-force is = γ(P´/c, F´), we have:



Let’s consider specifically the parts of the force parallel and perpendicular to the velocity.



This coheres with what we know from electrodynamics for instance. Consider a particle at rest, in an electric, magnetic field **E**, **B**. Then the force on it would be:



Now go to a frame moving backwards with velocity -**v**, so that our particle is moving with velocity **v**. Then the fields transform according to:



In our case, where B in the rest frame is zero, this simplifies to:



We can write this as:



Separating out the parallel and perpendicular parts, recalling **β** = **v**/c, we have:



So everything checks out! There is a useful identify to consider. Let’s form the tensor ∙**.**  We have:



Technically, I think this identity only holds for cases where the force doesn’t increase the rest mass in the rest frame. I read somewhere that nuclear forces do in fact increase the rest mass. And for these, therefore P doesn’t equal F∙v, because energy can increase without velocity increasing. But presuming we’re dealing with classical forces (EM, gravity at least), we have:



**Least action modified to accommodate special relativity**

As has been found in the past, having a least action formulation to the relativistic laws of physics does prove useful. So what Lagrangian would give the correct relativistically invariant equations of motion? We can verify that the following choice does the job.



which is as before, except that the KE has been replaced by a negative factor of the total particle energy. For instance, let’s write down the equations of motion, neglecting the EM field,



which is simply:



as required.

**Example**

Consider the motion of a particle under a constant force, in 1D, starting from rest. Then we have,



formulating the equation of motion we have,



Solving for v gets us,



Integrating again gives us:



Expanding for a small force we get,



which is as we’d expect. The Lagrangian formulation often makes the analysis a little easier though.

**Relativistically invariant least action principle**

The Lagrangian above is perfectly adequate, but a little asymmetric in the sense that it treats time and position in different ways. It would be nice if we could write a least action principle that treated time and space symmetrically, like the relativistically invariant formulation of N2L and such that we accomplished above. What would such a Lagrangian look like? Well it should be relativistically invariant scalar. And it should have something to do with the energy obviously. A scalar quantity that is related to the energy is magnitude of the energy-momentum space time vector, or relatedly the magnitude of the space-time velocity vector (could use either one in this case). So if we were to try to guess a form for the relativistically invariant Lagrangian we might formulate the following guess, neglecting external forces:



(could have added a (1/2)m factor in there to make it look more like kinetic energy, but that’s the *classical* expression, and photons don’t have mass anyway, so just leave as is) Let’s apply the Euler-Lagrange equations of motion to this entity,



And we see that this does indeed work, i.e. it gives the same equations as



in a force free environment.

**Photon Dynamics**

For photons, we cannot straightaway use N2L because photons do not experience a force per se´. Instead we just differentiate the displacement w/r to something else, like arc length perhaps (an affine parameter in general). We can just say that photons follow space-time geodesics, which is to say their path is defined by an unchanging tangent vector. And so we have:



FWIW (and it’s worth something), I believe we can replace the υ with p.